MATH 54 - HINTS TO HOMEWORK 8

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Here are a couple of hints to Homework 8! Enjoy :)

Note: For all those problems, to find the eigenvalues of A, calculate $det(\lambda I - A)$ (or $det(A - \lambda I)$ if you want), and find the zeros of the resulting polynomial. To find the eigenvectors of A, for each eigenvalue λ that you found, find a basis for $Nul(\lambda I - A)$ (or $Nul(A - \lambda I)$). Also, you should never get $Nul(A - \lambda I) = \{\mathbf{0}\}$

SECTION 5.1: EIGENVALUES AND EIGENVECTORS

5.1.3, 5.1.5. Calculate $A\mathbf{v}$, where A is the given matrix and $\mathbf{v}$ is the given vector.

5.1.9, 5.1.11, 5.1.16. All you have to do is find a basis for $Nul(\lambda I - A)$ for each eigenvalue λ you found! Remember, you should *never* get $Nul(\lambda I - A) = \{\mathbf{0}\}$. In that case, you probably made an algebra mistake!

5.1.17. Remember that the determinant of an upper-triangular matrix is just the product on the entries of the diagonal!

5.1.21.

(a) **F** ($\mathbf{x}$ has to be nonzero)
(b) **T** (IMT)
(c) **T**
(d) **T** (depending on what you mean by easy and hard :))
(e) **F** (that has nothing to do with eigenvalues, to find the eigenvalues, just find $det(\lambda I - A)$)

5.1.22.

(a) **F** ($\mathbf{x}$ has to be nonzero)
(b) **F** (Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Here $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent eigenvectors, but they correspond to the same eigenvalue $\lambda = 1$. The correct statement is that if $\lambda 1 \neq \lambda 2$ and $\mathbf{v_1}$ and $\mathbf{v_2}$ are eigenvectors corresponding to λ_1 and λ_2, then $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent)
(c) ?????? (what is a steady-state vector for a stochastic matrix???? Just write $\mathbf{T}$, and don't worry about it)

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(d) **F** (Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then one eigenvalue is $\lambda = 0$ because A is not invertible, but 0 is not on the diagonal of A. Or pick any other example, like the matrix in 5.1.11)

(e) **T** ($Nul(\lambda I - A)$)

5.1.24. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (eigenvalue $\lambda = 1$)

5.1.25. This is cute! Suppose $A\mathbf{v} = \lambda\mathbf{v}$, then multiply by A^{-1} to get $A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v}$, then $\mathbf{v} = A^{-1}(\lambda\mathbf{v}) = \lambda A^{-1}\mathbf{v}$, and divide by $\lambda \neq 0$ (since A is invertible) to get $\frac{1}{\lambda}\mathbf{v} = A^{-1}\mathbf{v}$, that is $\boxed{A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}}$

5.1.26. This is also cute! Suppose λ is an eigenvalue for A, then $A\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v}$. Apply A to this equation to get $AA\mathbf{v} = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$. But $AA = A^2 = O$, so $O\mathbf{v} = \lambda^2\mathbf{v}$, so $\lambda^2\mathbf{v} = \mathbf{0}$. But the only way this happens is if $\lambda^2 = 0$, so $\lambda = 0$

SECTION 5.2: THE CHARACTERISTIC EQUATION

5.2.1, 5.2.3, 5.2.9, 5.2.11, 5.2.13. Calculate $det(\lambda I - A)$ (or $det(A - \lambda I)$ if you want) and find the zeros of the resulting polynomial. To do this, it's wise to expand the determinant along a row with lots of 0s.

5.2.19. Just set $\lambda = 0$. Then $det(A - \lambda I)$ becomes $det(A)$.

5.2.21.

(a) **F**

(b) **F** (for example, multiplying a row by 2 doubles the determinant)

(c) **T**

(d) **F** (-5 is an eigenvalue)

5.2.24. If A and B are similar, then there exists a matrix P with $A = PBP^{-1}$. Then calculate $det(A)$.

SECTION 5.3: DIAGONALIZATION

5.3.1. If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$

5.3.7, 5.3.11, 5.3.17. This means: Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Here D is the diagonal matrix whose entries are the eigenvalues of A and P is the matrix whose columns are the corresponding eigenvectors. **Remember** that the eigenvalues are given to you on the previous page!!! Also, for 5.3.17, since it's a lower-triangular matrix, the eigenvalues are $\lambda = 4, 5$. To find the eigenvectors, find a basis for $Nul(\lambda I - A)$, where λ is the given eigenvalue.

5.3.21.

(a) **T** (by definition)
(b) **T**
(c) **F** (The 'only if' part is true, but the 'if'-part is false! For example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eigenvalue $\lambda = 1$ (so 2 eigenvalues counting multiplicities), but is not diagonalizable)
(d) **F** (Consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is diagonalizable (in fact a *diagonal* matrix), but not invertible)

5.3.22.

(a) **F** (n *linearly independent* eigenvectors)
(b) **F** ($A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable (in fact diagonal), but it has only 1 eigenvalue $\lambda = 1$)
(c) **T** (Suppose for simplicity that $P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Then $AP = A \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} A\mathbf{v_1} & A\mathbf{v_2} \end{bmatrix}$ and $PD = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v_1} & \lambda_2\mathbf{v_2} \end{bmatrix}$. However $AP = PD$, so $\begin{bmatrix} A\mathbf{v_1} & A\mathbf{v_2} \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v_1} & \lambda_2\mathbf{v_2} \end{bmatrix}$ and we get $A\mathbf{v_1} = \lambda_1\mathbf{v_1}$ and $A\mathbf{v_2} = \lambda_2\mathbf{v_2}$)
(d) **F** (Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It is invertible, but not diagonalizable)

5.3.31. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It is invertible, but not diagonalizable. To show it is not diagonalizable, first the eigenvectors of A. If A were diagonalizable, then the eigenspace of A would be $2-$dimensional, but here it is only $1-$dimensional!

5.3.32. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is diagonalizable (in fact a *diagonal* matrix), but not invertible

SECTION 5.4: EIGENVECTORS AND LINEAR TRANSFORMATIONS

5.4.19. To show that B is invertible, just calculate $det(B)$ (we know $det(A) \neq 0$ since A is invertible). The matrix Q in question is just P! To show this, just calculate $B^{-1} = (P^{-1}AP)^{-1}$!

5.4.20. Since A is similar to B, $A = PBP^{-1}$, but then $A^2 = PB\cancel{P^{-1}}\cancel{P}BP^{-1} = PB^2P^{-1}$, so A^2 is similar to B^2

5.4.21. By assumption, we know $B = PAP^{-1}$ and $C = QAQ^{-1}$. But then $A = Q^{-1}CQ$, so $B = PAP^{-1} = P(Q^{-1}CQ)P^{-1} = (PQ^{-1})C(PQ^{-1})^{-1} = \widetilde{P}C\widetilde{P}^{-1}$, where $\widetilde{P} = PQ^{-1}$.

5.4.26. Since A is diagonalizable, $A = PDP^{-1}$. Then use the fact that if B is invertible, then $Tr(AB) = Tr(A)$ and $Tr(BA) = A$. That gives $Tr(A) = Tr(PDP^{-1}) = Tr(DP^{-1}) = Tr(D)$. But $Tr(D)$ is just the sum of the diagonal entries of D, i.e. the sum of the eigenvalues of D.

SECTION 5.5: COMPLEX EIGENVALUES

5.5.1, 5.5.5. Use the same technique you usually use to find eigenvalues and eigenvectors, i.e. find $det(\lambda I - A)$ and for every eigenvalue λ you found, find a basis for $Nul(\lambda I - A)$. Here you only have to do half of the work, because if $\mathbf{v}$ is a (complex) eigenvector corresponding to λ, then $\bar{\mathbf{v}}$ is an eigenvector corresponding to $\bar{\lambda}$.

For example, if one eigenvalue is $\lambda = 1 + 3i$, and $\mathbf{v} = \begin{bmatrix} 2+4i \\ 3+6i \end{bmatrix}$, then another eigenvalue is $\bar{\lambda} = 1 - 3i$ and another eigenvector is $\bar{\mathbf{v}} = \begin{bmatrix} 2-4i \\ 3-6i \end{bmatrix}$